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MASTER THESIS

# Maximum-distance Race Strategies for a Fully Electric Endurance Race Car 

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# Maximum-distance Race Strategies for a Fully Electric Endurance Race Car 

Jorn van Kampen


#### Abstract

This thesis presents a bi-level optimization framework to compute the maximum-distance stint and charging strategies for a fully electric endurance race car, while accounting for thermal limitations of the powertrain components. Thereby, the lower level computes the minimum-stint-time Powertrain Operation (PO) for a given charge time and stint length, whilst the upper level leverages that information to jointly optimize the stint length, charge time and number of pit stops, in order to maximize the driven distance in the course of a fixed-time endurance race. Specifically, we first extend a convex lap time optimization framework to capture multiple laps and thermal models, and use it to create a map linking the charge time and stint length to the achievable stint time. Second, we leverage the map to frame the maximum-race-distance problem as a mixed-integer second order conic program that can be efficiently solved to the global optimum with off-the-shelf optimization algorithms. Finally, we showcase our framework on a 6 h race around the Zandvoort circuit. Our results show that the optimal race strategy can involve partially charging the battery, and that, compared to the case where the stints are optimized for a fixed number of pit stops, jointly optimizing the stints and number of pit stops can significantly increase the driven distance of several laps.


## I. InTRODUCTION

THE ELECTRIFICATION of race cars has been increasing in popularity over the last years, owing to the advent of hybrid electric Formula 1 cars and Le Mans Hypercars, and battery electric vehicles in Formula E. In a setting where every millisecond counts, it is of paramount importance to make efficient use of the energy stored on-board via optimized Energy Management Strategy (EMS), whilst respecting the thermal limits of the powertrain components. In this context, the possibility of recharging the battery in the course of the race further complicates the problem, requiring race engineers to strike the best trade-off between reducing consumptions and pit-stops at the cost of lap-time, or driving faster with more pit-stops, whilst avoiding damage to the powertrain components by staying within the thermal limits. This conflict is particularly important in endurance racing, where the objective is to maximize the driven distance in a fixed amount of time, which can range up to 24 h [1]. In this setting, the car has to be strategically recharged during pit stops in order to maintain a competitive performance and maximize the distance driven. This calls for algorithms to compute the maximum-distance race strategies in terms of number of pit stops, stint length and charge time (which is directly correlated to charged energy), accounting for the optimal stint strategies in terms of energy management, thermal management and Powertrain Operation (PO). Against this backdrop, this thesis presents a bi-level optimization framework to compute the maximum-distance race strategies with global optimality guarantees.


Fig. 1. InMotion's fully electric endurance race car.


Fig. 2. Schematic layout of the electric race car powertrain topology consisting of a battery (BAT), inverter (INV), electric machine (EM) and final drive (FD). The arrows indicate positive power flows.

Related Literature: This work pertains to two main research streams: single-lap optimization of the EMSs jointly with the vehicle trajectory or for a given race line, and fullrace optimization via simulations.

Several authors optimized the minimum-lap-time race line for a single race lap using both direct and indirect optimization methods [2]-[8]. Some of these studies also include a maximum energy consumption per lap to approach racing conditions [9]. Similar approaches extend the minimum-lap-time problems to minimum-race-time problems. They consider temperature dynamics, and optimize for multiple consecutive race laps to enable a variable amount of energy consumed per lap, but formulate the optimization problem in space domain for an a-priori-known number of laps [10], [11]. Finally, considering the race line to be fixed, multilap EMSs are optimized, leveraging nonlinear optimization techniques [12] or artificial neural networks [13]. However, these papers lack global optimality guarantees.

Against this backdrop, assuming the race line to be available in the form of a maximum speed profile, convex optimization has been successfully leveraged to compute the globally optimal EMSs for hybrid and fully electric race vehicles [14], [15], also including gear shift strategies [16], different transmission technologies [17] and thermal limitations [18]. Yet these methods are focused on singlelap problems and do not capture pit-stops and recharging processes. In addition, the studies focused on thermal limitations leverage an iterative algorithm, thereby losing global optimality guarantees when the velocity is jointly optimized.

The second relevant research stream involves race sim-
ulations, in which entire races are optimized on a per lap basis [19], [20]. However, these studies mainly focus on optimal tire strategies by modeling their degradation as a lap time increase and do not capture the charging and PO strategies.

In conclusion, to the best of the authors' knowledge, there are no methods specifically focusing on race strategies in endurance scenarios, whereby the single-stint operational strategies are jointly optimized and account for the thermal limits of the powertrain components.

Statement of Contributions: This thesis presents a bi-level mixed-integer convex optimization framework to efficiently compute the globally optimal, maximum-distance endurance race strategies and the corresponding PO in the individual stints. Our low-level algorithm computes the optimal stint time for a given number of laps and different levels of recharged battery energy. To preserve convexity, we describe the electric motor (EM) efficiency by using speed-dependent in- and output forces. Subsequently, we fit the relationship between the stint length, the charged energy, and the achievable stint time as a second-order conic constraint, which we leverage in the high-level algorithm. Thereby we frame the maximum-distance race problem as a mixed-integer secondorder conic program which jointly optimizes the stint length, the charge time-i.e., the charge energy-and the number of pit stops. The resulting problem can be rapidly solved with off-the-shelf numerical solvers with global optimality guarantees. A preliminary version of this thesis was published at the 2022 European Control Conference [21]. In this extended version, we include a battery loss model that captures the dependence on its energy and temperature and identify a method to model the battery and EM temperature dynamics in a convex form. Moreover, we leverage a convex framework to directly include the vehicle dynamics in the form of a single-track model in the low-level control problem, so that we no longer rely on a pre-computed maximum speed profile and reformulate the high-level control problem to account for the aforementioned extensions. Finally, we showcase our framework on the Zandvoort circuit, highlighting the importance of jointly optimizing the number of pit stops with the number of laps and charging strategies.

Organization: The remainder of this thesis is structured as follows: Section II presents the minimum-stint-time control problem, after which Section III frames the maximum-racedistance control problem. We discuss some of the limitations of our work in Section IV and showcase our framework for a 6 h race in Section V. Finally, Section VI draws the conclusions and provides an outlook on future research.

## II. Low-Level Stint Optimization

This section illustrates the minimum-stint-time control problem in space domain, since minimizing the stint time given a fixed distance represents the dual problem of maximizing distance within a fixed time. We leverage an existing convex framework [22], reformulated to a single-track model without steering and side-slip angles, as we do not consider torque-vectoring, and extend it to allow multi-lap optimization, whilst improving the EM and battery model accuracy to
include temperature dynamics. From the time-optimal control problem, we obtain the minimum stint time for a given stint length and charge time (which can be equivalently expressed in terms of available battery energy).

Fig. 2 shows a schematic representation of the powertrain topology of the electric race car. The EM propels both of the rear wheels through a fixed final drive (FD), while receiving energy from the battery pack via the inverter. As with most electric vehicles, the EM can also operate as a generator, thus we account for a bi-directional energy flow between the battery and the wheels. In addition, we consider auxiliary components that are powered from the main battery as a uni-directional energy flow.

In reality, the driver controls the EM torque through the accelerator pedal and as such we define the mechanical EM power $P_{\mathrm{m}}$ as the input variable. As state variables, we choose the battery energy $E_{\mathrm{b}}$, battery temperature $\vartheta_{\mathrm{b}}$, EM temperature $\vartheta_{\mathrm{m}}$ and the kinetic energy of the vehicle $E_{\text {kin }}$. The remaining energy flows between the powertrain components are the propulsion power $P_{\mathrm{p}}$, electrical EM power $P_{\mathrm{ac}}$, electrical inverter power $P_{\mathrm{dc}}$ and auxiliary supply $P_{\text {aux }}$. Since we formulate the control problem in space domain, we ultimately define the model in terms of forces rather than power. Thus we divide power by the vehicle velocity, since the space-derivative of energy is defined with respect to the vehicle.

## A. Objective and Longitudinal Dynamics

In racing, the objective is to minimize the lap times over the entire race. Since we only consider a stint in the lowlevel control problem, the objective is to minimize the stint time $t_{\text {stint }}$, which is defined as

$$
\begin{equation*}
\min t_{\mathrm{stint}}=\min \int_{0}^{S_{\mathrm{stint}}} \frac{\mathrm{~d} t}{\mathrm{~d} s}(s) \mathrm{d} s \tag{1}
\end{equation*}
$$

where $S_{\text {stint }}$ is the stint length in terms of distance and $\frac{\mathrm{d} t}{\mathrm{~d} s}(s)$ is the lethargy, which is the inverse of the vehicle velocity $v(s) \geq 0$. To implement the lethargy as a convex constraint, we define

$$
\begin{equation*}
\frac{\mathrm{d} t}{\mathrm{~d} s}(s) \geq \frac{1}{v(s)} \tag{2}
\end{equation*}
$$

which is a convex relaxation that holds with equality in case of an optimal solution [14].

In this study, we limit ourselves to non-torque-vectoring powertrain topologies and thereby only capture the longitudinal vehicle dynamics with the use of a bicycle model that models the front and rear axle individually. The longitudinal force balance is given by

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} s} E_{\mathrm{kin}}(s)=F_{\mathrm{x}, \mathrm{~F}}(s)+F_{\mathrm{x}, \mathrm{R}}(s)-F_{\mathrm{drag}}(s)-m \cdot g \cdot \sin (\theta(s)) \tag{3}
\end{equation*}
$$

where $F_{\mathbf{x}, i}(s)$ is the longitudinal force per axle with $i \in$ $[\mathrm{F}, \mathrm{R}]$ denoting the front and rear axle respectively, $F_{\mathrm{drag}}(s)$ is the aerodynamic drag force, $m$ is the total mass of the vehicle, $g$ is the gravitational constant and $\theta(s)$ is the inclination of the track. The aerodynamic drag force is given by

$$
\begin{equation*}
F_{\mathrm{drag}}(s)=\frac{c_{\mathrm{d}} \cdot A_{\mathrm{f}} \cdot \rho}{m} \cdot E_{\mathrm{kin}}(s), \tag{4}
\end{equation*}
$$

where $c_{\mathrm{d}}$ is the drag coefficient, $A_{\mathrm{f}}$ is the frontal area of the vehicle and $\rho$ is the air density The longitudinal axle force are defined as

$$
\begin{equation*}
F_{\mathrm{x}, i}(s)=F_{\mathrm{p}, i}(s)-c_{\mathrm{r}} \cdot F_{\mathrm{z}, i}(s)-F_{\mathrm{brake}, i}(s), \tag{5}
\end{equation*}
$$

where $F_{\mathrm{p}, i}(s)$ is the propulsion force per axle, $c_{\mathrm{r}}$ is the rolling resistance coefficient, $F_{\mathrm{z}, i}(s)$ represents the vertical axle force and $F_{\text {brake }, i}(s)$ is the force from the mechanical brakes per axle. For non-driven wheels, we set $F_{\mathrm{p}, i}(s)=0$, whereas for driven wheels, we write (5) as two inequality constraints to capture the final drive losses through

$$
\begin{align*}
& F_{\mathrm{x}, i}(s) \leq F_{\mathrm{m}}(s) \cdot \eta_{\mathrm{fd}}-c_{\mathrm{r}} \cdot F_{\mathrm{z}, i}(s)-F_{\mathrm{brake}, i}(s)  \tag{6}\\
& F_{\mathrm{x}, i}(s) \leq F_{\mathrm{m}}(s) \cdot \frac{1}{\eta_{\mathrm{fd}}}-c_{\mathrm{r}} \cdot F_{\mathrm{z}, i}(s)-F_{\mathrm{brake}, i}(s), \tag{7}
\end{align*}
$$

where $F_{\mathrm{m}}(s)$ is the mechanical output force from the EM and $\eta_{\mathrm{fd}}$ is the efficiency of the final drive, assumed constant. Due to the objective (1), in case of traction, (6) will hold with equality, whilst in case of regenerative braking, (7) will hold with equality, thus capturing the bi-directional power flow.

The lateral force balance is defined as

$$
\begin{equation*}
2 E_{\mathrm{kin}}(s) \cdot R_{\mathrm{inv}}(s)=F_{\mathrm{y}, \mathrm{~F}}(s)+F_{\mathrm{y}, \mathrm{R}}(s) \tag{8}
\end{equation*}
$$

where $R_{\text {inv }}(s)$ is the pre-computed inverse corner radius of the track and $F_{\mathrm{y}, i}(s)$ represents the lateral force per axle.

The vertical force balance consists of the static load and the aerodynamic downforce as

$$
\begin{equation*}
F_{\mathrm{z}, \mathrm{~F}}(s)+F_{\mathrm{z}, \mathrm{R}}(s)=m \cdot g \cdot \cos (\theta(s))+F_{\text {down }}(s), \tag{9}
\end{equation*}
$$

where $F_{\mathrm{z}, i}(s)$ is the vertical force per axle and $F_{\text {down }}(s)$ is the aerodynamic downforce given by

$$
\begin{equation*}
F_{\text {down }}(s)=\frac{c_{1} \cdot A_{\mathrm{f}} \cdot \rho}{m} \cdot E_{\mathrm{kin}}(s) \tag{10}
\end{equation*}
$$

where $c_{1}$ is the lift coefficient. We consider steady-state cornering only, thereby assuming a yaw moment equilibrium given by

$$
\begin{equation*}
F_{\mathrm{y}, \mathrm{~F}}(s) \cdot l_{\mathrm{F}}=F_{\mathrm{y}, \mathrm{R}}(s) \cdot l_{\mathrm{R}}, \tag{11}
\end{equation*}
$$

where $l_{i}$ represents the horizontal distance from the axle to the center of gravity (CoG).

The longitudinal load transfer is determined through the pitch moment equilibrium, which is defined by

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} s} E_{\mathrm{kin}}(s) \cdot h_{\mathrm{G}}= & F_{\mathrm{z}, \mathrm{R}}(s) \cdot l_{\mathrm{R}}-F_{\mathrm{z}, \mathrm{~F}}(s) \cdot l_{\mathrm{F}} \\
& \quad-F_{\mathrm{drag}}(s) \cdot h_{\mathrm{P}}-F_{\mathrm{down}}(s) \cdot l_{\mathrm{GP}} \tag{12}
\end{align*}
$$

where $h_{\mathrm{G}}$ is the height of the CoG with respect to the ground, $h_{\mathrm{P}}$ is the height of the center of pressure ( CoP ) with respect to the ground and $l_{\mathrm{GP}}$ is the horizontal distance from the CoG to the CoP.

The longitudinal and lateral forces are bounded by their respective friction circles per axle, which are defined by the convex set written as

$$
\begin{equation*}
F_{\mathrm{x}, i}^{2}(s)+F_{\mathrm{y}, i}^{2}(s) \leq\left(\mu_{i} \cdot F_{\mathrm{z}, i}(s)\right)^{2} \tag{13}
\end{equation*}
$$

where $\mu_{i}$ represents the tire friction coefficient, assumed constant. Although the constraint function is not convex, it specifies a convex set, which is shown in the Appendix.

The majority of racing vehicles are equipped with mechanical brakes that provide a fixed brake force ratio between the front and rear wheels. Therefore, we define a relation between the front and rear brake force as

$$
\begin{equation*}
F_{\text {brake }, \mathrm{R}}(s) \cdot \delta_{\text {brake }}=F_{\text {brake }, \mathrm{F}}(s) \cdot\left(1-\delta_{\text {brake }}\right), \tag{14}
\end{equation*}
$$

where $\delta_{\text {brake }}$ represents the brake balance with respect to the front, which is assumed to remain constant during the race.

The relation between the kinetic energy and velocity of the vehicle is defined by a convex relaxation as

$$
\begin{equation*}
E_{\text {kin }}(s) \geq \frac{1}{2} \cdot m \cdot v^{2}(s) \tag{15}
\end{equation*}
$$

In contrast to single-lap scenarios, a stint is represented by the vehicle starting and stopping at the pit box with a certain number of free-flow laps in between. However, since we are working in space domain, the lethargy would diverge to infinity for zero velocity. To solve this issue, we define a minimal velocity $v_{\text {min }}$ close to standstill and enforce this value to the initial and final velocity with

$$
\begin{equation*}
E_{\text {kin }}(0)=E_{\text {kin }}\left(S_{\mathrm{stint}}\right)=\frac{1}{2} \cdot m \cdot v_{\min }^{2} \tag{16}
\end{equation*}
$$

Finally, the vehicle should adhere to a strict speed limit, of which the exact value is track-dependent, when driving through the pit lane. Therefore, we define an upper bound $v_{\text {pit, max }}$ on the vehicle velocity when the vehicle is exiting or entering the pit as

$$
\begin{equation*}
E_{\mathrm{kin}}(s) \leq \frac{1}{2} \cdot m \cdot v_{\mathrm{pit}, \max }^{2} \quad \forall s \in \mathcal{S}_{\mathrm{pit}} \tag{17}
\end{equation*}
$$

where $\mathcal{S}_{\text {pit }}$ is the set of distance-based positions that are part of the pit lane.

## B. Electric Machine

This section derives a convex representation of the operating limits and power losses of the EM. Moreover, we derive the thermal dynamics of the EM and compare the model against real-world test data.

In general, we can distinguish between a maximum torque and maximum power operating region for an EM. Translating this to constraints in space domain results in a lower and upper bound on the mechanical output force of the EM for the maximum torque region as

$$
\begin{equation*}
F_{\mathrm{m}}(s) \in\left[-\frac{T_{\mathrm{m}, \max } \cdot \gamma_{\mathrm{fd}}}{r_{\mathrm{w}}}, \frac{T_{\mathrm{m}, \max } \cdot \gamma_{\mathrm{fd}}}{r_{\mathrm{w}}}\right] \tag{18}
\end{equation*}
$$

where $T_{\mathrm{m}, \text { max }}$ is the maximum torque the EM can deliver, $\gamma_{\mathrm{fd}}$ is the final drive ratio and $r_{\mathrm{w}}$ is the radius of the rear wheels. Note that we include the final drive ratio, as we define the space-derivatives with respect to the vehicle reference frame. Similarly, the mechanical output force of the EM within the maximum power region is bounded as

$$
\begin{equation*}
F_{\mathrm{m}}(s) \in\left[-P_{\mathrm{m}, \max } \cdot \frac{\mathrm{~d} t}{\mathrm{~d} s}(s), P_{\mathrm{m}, \max } \cdot \frac{\mathrm{~d} t}{\mathrm{~d} s}(s)\right] \tag{19}
\end{equation*}
$$

where $P_{\mathrm{m}, \text { max }}$ is the maximum power the EM can deliver.
We model the EM force losses $F_{\mathrm{m}, \mathrm{l}}(s)$ rather than the power losses as a function of the vehicle velocity and force of the EM. In general, an EM efficiency map shows large losses at low rotational velocities. Therefore, we want to include a term in our losses fit that is inversely proportional


Fig. 3. A speed- and torque-dependent model of the EM. The normalized RMSE of the model is $1.49 \%$ w.r.t. the maximum motor input force $F_{\mathrm{ac}}$.
to the vehicle velocity. To ensure convexity, we model the EM losses as

$$
\begin{equation*}
F_{\mathrm{m}, 1}(s)=x_{\mathrm{m}, 1}^{\top}(s) Q_{\mathrm{m}, 1} x_{\mathrm{m}, 1}(s) \tag{20}
\end{equation*}
$$

where $x_{\mathrm{m}, 1}(s)=\left[\frac{1}{\sqrt{v(s)}} \sqrt{v(s)} \frac{F_{\mathrm{m}}(s)}{\sqrt{v(s)}}\right]^{\top}$ and $Q_{\mathrm{m}, 1} \in \mathbb{S}_{+}^{3}$ is a symmetric positive semi-definite matrix of coefficients, whose values are determined through semi-definite programming. Fig. 3 shows the accuracy of our model. To implement the losses in a convex manner, we take the relation of the electrical EM input force $F_{\mathrm{ac}}(s)$ to the mechanical output force as

$$
\begin{equation*}
F_{\mathrm{ac}}(s)=F_{\mathrm{m}}(s)+F_{\mathrm{m}, 1}(s), \tag{21}
\end{equation*}
$$

substitute the loss model, relax it and rewrite to a relaxation describing a convex set as

$$
\begin{equation*}
\left(F_{\mathrm{ac}}(s)-F_{\mathrm{m}}(s)\right) \cdot v(s) \geq y_{\mathrm{m}, 1}^{\top}(s) Q_{\mathrm{m}, 1} y_{\mathrm{m}, 1}(s) \tag{22}
\end{equation*}
$$

where $y_{\mathrm{m}, 1}(s)=\left[1 v(s) F_{\mathrm{m}}(s)\right]^{\top}$. The convexity of this constraint is shown in the Appendix by writing it as a secondorder conic constraint.

For the cooling circuit of the EM, we consider a conventional setup using liquid cooling and radiators, as commonly applied in motorsport. The losses are assumed to be converted to heat, thereby changing the EM temperature according to the first-order temperature ordinary differential equation (ODE) given by

$$
\begin{equation*}
C_{\mathrm{m}} \cdot \frac{\mathrm{~d} \vartheta_{\mathrm{m}}}{\mathrm{~d} t}(s)=P_{\mathrm{m}, 1}(s)-P_{\mathrm{m}, \mathrm{c}}(s) \tag{23}
\end{equation*}
$$

where $C_{\mathrm{m}}$ is the total lumped thermal capacity of the EM, $\vartheta_{\mathrm{m}}(s)$ is the temperature of the EM, $P_{\mathrm{m}, 1}(s)$ are the EM power losses and $P_{\mathrm{m}, \mathrm{c}}(s) \geq 0$ represents the power outflow to the cooling liquid as

$$
\begin{equation*}
P_{\mathrm{m}, \mathrm{c}}(s)=\frac{\vartheta_{\mathrm{m}}(s)-\vartheta_{\mathrm{m}, \mathrm{c}}}{\kappa_{\mathrm{m}}} \tag{24}
\end{equation*}
$$

where $\vartheta_{\mathrm{m}, \mathrm{c}}$ represents the temperature of the cooling liquid and $\kappa_{\mathrm{m}}$ is the thermal resistance between the EM and the cooling liquid, where we assume both parameters to be constant. Rewriting (23) to space domain results in

$$
\begin{equation*}
C_{\mathrm{m}} \cdot \frac{\mathrm{~d} \vartheta_{\mathrm{m}}}{\mathrm{~d} s}(s)=F_{\mathrm{m}, 1}(s)-F_{\mathrm{m}, \mathrm{c}}(s) \tag{25}
\end{equation*}
$$



Fig. 4. Semi-definite fit of the EM temperature together with the relative error. The normalized RMSE is $0.15 \%$ w.r.t. the maximum temperature.
where $F_{\mathrm{m}, \mathrm{c}}(s) \geq 0$ is the force-equivalence of the EM cooling power. This EM cooling force is ultimately obtained by rewriting (24) to forces using a convex relaxation and linear equality constraint. To obtain a convex representation, we approximate the EM temperature as

$$
\begin{equation*}
\bar{\vartheta}_{\mathrm{m}}(s)=x_{\mathrm{m}, \vartheta}^{\top}(s) Q_{\mathrm{m}, \vartheta} x_{\mathrm{m}, \vartheta}(s)+\vartheta_{\mathrm{m}, 0} \tag{26}
\end{equation*}
$$

where $x_{\mathrm{m}, \vartheta}(s)=\left[1 \vartheta_{\mathrm{m}}(s)\right]^{\top}, \vartheta_{\mathrm{m}, 0}$ is an offset required to obtain positive values and $Q_{\mathrm{m}, \vartheta} \in \mathbb{S}_{-}^{2}$ is a negative semidefinite matrix of coefficients, obtained through semi-definite programming. The semi-definite fit of the temperature together with the relative error is shown in Fig. 4. We select a negative semi-definite matrix, since we require an upper bound on the EM temperature. Translating (24) to forces and substituting the offset of (26) results in

$$
\begin{equation*}
F_{\mathrm{m}, \mathrm{c}}(s)=\frac{\bar{F}_{\mathrm{m}, \mathrm{c}}(s)+\left(\vartheta_{\mathrm{m}, 0}-\vartheta_{\mathrm{m}, \mathrm{c}}\right) \cdot \frac{\mathrm{d} t}{\mathrm{~d} s}(s)}{\kappa_{\mathrm{m}}} \tag{27}
\end{equation*}
$$

where $\bar{F}_{\mathrm{m}, \mathrm{c}}(s)$ is an intermediate variable used to obtain a convex formulation through

$$
\begin{equation*}
\bar{F}_{\mathrm{m}, \mathrm{c}}(s) \cdot v(s) \leq x_{\mathrm{m}, \vartheta}^{\top}(s) Q_{\mathrm{m}, \vartheta} x_{\mathrm{m}, \vartheta}(s) \tag{28}
\end{equation*}
$$

which can be written as a second-order conic constraint (see Appendix). Since the cooling circuit consists of liquid coolant flowing through the EM to a set of radiators, the coolant temperature cannot exceed the EM temperature, resulting in

$$
\begin{equation*}
F_{\mathrm{m}, \mathrm{c}}(s) \geq 0 \tag{29}
\end{equation*}
$$

To obtain the thermal parameters of the EM, we apply the convex loss- and cooling model to a combination of data sets recorded from vehicle telemetry. Fig. 5 shows a comparison between the EM temperature from one of the data sets and the thermal model.

To prevent the EM from overheating, we define an upper bound on the temperature through

$$
\begin{equation*}
\vartheta_{\mathrm{m}}(s) \leq \vartheta_{\mathrm{m}, \max } \tag{30}
\end{equation*}
$$

where $\vartheta_{\mathrm{m}, \text { max }}$ is the maximum temperature of the EM. Lastly, we specify an initial value for the EM temperature as

$$
\begin{equation*}
\vartheta_{\mathrm{m}}(0)=\vartheta_{\mathrm{m}, \mathrm{init}} \tag{31}
\end{equation*}
$$

where $\vartheta_{\mathrm{m} \text {,init }}$ is the initial value for the temperature and is calculated during pre-processing using a lookup table that has the charge time as an input.


Fig. 5. Comparison between the thermal model and one of the vehicle telemetry data samples. In total, three data sets were used to fit the model parameters, of which the average normalized RMSE was $4.12 \%$ w.r.t. the maximum temperature of each data set.

## C. Inverter

In this section, we derive a quadratic model for the inverter losses. As opposed to the EM, we do not model the inverter temperature, since we assume that the motor-inverter combination is designed such that the EM is thermally limiting. We apply the general quadratic power loss model of the form

$$
\begin{equation*}
P_{\mathrm{dc}}(s)=\alpha \cdot P_{\mathrm{ac}}^{2}(s)+P_{\mathrm{ac}}(s), \tag{32}
\end{equation*}
$$

where $\alpha$ is an efficiency parameter, subject to identification. Converting this constraint to forces, rewriting and relaxing results in

$$
\begin{equation*}
\left(F_{\mathrm{dc}}(s)-F_{\mathrm{ac}}(s)\right) \cdot \frac{\mathrm{d} t}{\mathrm{~d} s}(s) \geq \alpha \cdot F_{\mathrm{ac}}^{2}(s) \tag{33}
\end{equation*}
$$

where $F_{\mathrm{dc}}(s)$ is the force equivalent to the electrical inverter power. The convexity of this constraint is shown in the Appendix by writing it as a second-order conic constraint.

## D. Battery

This section derives a model for the battery efficiency and the power-split between the electrical inverter power and the auxiliary component power. The latter can be observed from Fig. 2 and is written as

$$
\begin{equation*}
P_{\mathrm{b}}(s)=P_{\mathrm{dc}}(s)+P_{\mathrm{aux}}, \tag{34}
\end{equation*}
$$

where $P_{\mathrm{b}}(s)$ is the battery power at the terminals. Here, the auxiliary component supply is assumed to be constant and uni-directional, while the other powers are bi-directional. Converting (34) to forces results in

$$
\begin{equation*}
F_{\mathrm{b}}(s)=F_{\mathrm{dc}}(s)+P_{\mathrm{aux}} \cdot \frac{\mathrm{~d} t}{\mathrm{~d} s}(s) \tag{35}
\end{equation*}
$$

where $F_{\mathrm{b}}(s)$ is the force equivalent of the battery power at the terminals.

The battery efficiency is mostly determined by its internal resistance $R_{0}$ and open-circuit voltage $V_{\text {oc }}$. We derive the battery losses $P_{\mathrm{b}, 1}\left(E_{\mathrm{b}}, \vartheta_{\mathrm{b}}, P_{\mathrm{i}}\right)$ from a Thévenin model [23] as

$$
\begin{equation*}
P_{\mathrm{b}, 1}\left(E_{\mathrm{b}}, \vartheta_{\mathrm{b}}, P_{\mathrm{i}}\right)=\frac{1}{P_{\mathrm{sc}}\left(E_{\mathrm{b}}, \vartheta_{\mathrm{b}}\right)} \cdot P_{\mathrm{i}}^{2}(s) \tag{36}
\end{equation*}
$$

where $P_{\mathrm{sc}}\left(E_{\mathrm{b}}, \vartheta_{\mathrm{b}}\right)=\frac{V_{\mathrm{oc}}^{2}\left(E_{\mathrm{b}}\right)}{R_{0}\left(E_{\mathrm{b}}, \vartheta_{\mathrm{b}}\right)}$ is the short-circuit power [24]. In reality, both the internal resistance and opencircuit voltage are a function of the battery temperature and energy. However, since the influence of the battery temperature on the open-circuit voltage is rather small within


Fig. 6. A temperature- and energy-dependent battery model. The normalized RMSE of the model is $2.23 \%$ w.r.t. the maximum short-circuit power.
the operating window in racing scenarios, we neglect the dependency of the open-circuit voltage on temperature [25], [26]. For the dependency of internal resistance on temperature, we apply a correction factor inversely proportional to temperature [27], [28] to obtain

$$
\begin{equation*}
P_{\mathrm{sc}}\left(E_{\mathrm{b}}, \vartheta_{\mathrm{b}}\right)=\frac{V_{\mathrm{oc}}^{2}\left(E_{\mathrm{b}}\right)}{R_{0}\left(E_{\mathrm{b}}\right)} \cdot \frac{\vartheta_{\mathrm{b}}(s)}{\vartheta_{\mathrm{ref}}}, \tag{37}
\end{equation*}
$$

where $\vartheta_{\text {ref }}$ represents the reference temperature at which the battery data is measured. Similarly to the thermal EM model, we fit the short-circuit power in a convex manner through

$$
\begin{equation*}
P_{\mathrm{sc}}(s)=x_{\mathrm{b}, 1}^{\top}(s) Q_{\mathrm{b}, 1} x_{\mathrm{b}, 1}(s)+P_{\mathrm{sc}, 0} \tag{38}
\end{equation*}
$$

where $x_{\mathrm{b}, 1}(s)=\left[1 E_{\mathrm{b}}(s) \vartheta_{\mathrm{b}}(s)\right]^{\top}, P_{\mathrm{sc}, 0}$ is an offset required to obtain positive values and $Q_{\mathrm{b}, 1} \in \mathbb{S}_{-}^{2}$ is a negative semi-definite matrix of coefficients, identified through semidefinite programming. Again, we select a negative semidefinite matrix, since it is optimal to maximize the shortcircuit force and thereby we require an upper bound. The temperature- and energy-dependent model of the short-circuit power is shown in Fig. 6. Translating (37) to forces and substituting the offset of (38) results in

$$
\begin{equation*}
F_{\mathrm{sc}}(s) \cdot v(s)=\bar{F}_{\mathrm{sc}}(s)+P_{\mathrm{sc}, 0} \cdot \frac{\mathrm{~d} t}{\mathrm{~d} s}(s) \tag{39}
\end{equation*}
$$

where $F_{\mathrm{sc}}(s)$ is the short-circuit force and $\bar{F}_{\mathrm{sc}}(s)$ is an intermediate variable used to obtain a convex formulation through

$$
\begin{equation*}
\bar{F}_{\mathrm{sc}}(s) \cdot v(s) \leq x_{\mathrm{b}, 1}^{\top}(s) Q_{\mathrm{b}, 1} x_{\mathrm{b}, 1}(s) \tag{40}
\end{equation*}
$$

To obtain the battery losses during discharging $F_{\mathrm{b}, 1}(s)$, we translate (36) to forces and relax it, which results in

$$
\begin{equation*}
F_{\mathrm{b}, 1}(s) \cdot F_{\mathrm{sc}}(s) \geq F_{\mathrm{i}}^{2}(s) \tag{41}
\end{equation*}
$$

where $F_{\mathrm{i}}(s)$ is the internal battery force, which ultimately dictates a change in battery energy. To prevent the battery losses and short-circuit force from cooling the battery, we explicitly define

$$
\begin{align*}
F_{\mathrm{b}, 1}(s) & \geq 0  \tag{42}\\
F_{\mathrm{sc}}(s) & \geq 0 \tag{43}
\end{align*}
$$

In general, a Li-ion battery generates more heat during charging compared to discharging. Therefore, we add an additional term to the battery losses to represent the additional losses during charging. Since this term should only be present during negative power flow, we implement a set of inequality constraints similar to the final drive losses as

$$
\begin{align*}
& F_{\mathrm{i}}(s) \geq \quad F_{\mathrm{b}}(s)+F_{\mathrm{b}, 1}(s),  \tag{44}\\
& F_{\mathrm{i}}(s) \geq\left(1-\alpha_{\text {charge }}\right) \cdot F_{\mathrm{b}}(s)+F_{\mathrm{b}, 1}(s), \tag{45}
\end{align*}
$$

where $F_{\mathrm{b}}(s)$ is the battery force at the terminals and $\alpha_{\text {charge }} \geq 0$ is a coefficient that represents the additional charging losses. In energy-limited scenarios, (44) will hold with equality during discharging, whereas (45) will hold with equality during charging.

In contrast to the EM cooling, where the difference between the EM temperature and the ambient temperature is sufficient to apply radiators, the difference between the battery temperature and the ambient air is relatively small. Therefore, it is common to apply a refrigerant circuit instead of radiators to cool the battery during fast-charging pit stops and driving, which allows the coolant temperature to drop below the ambient level. Again, all losses are assumed to be converted to heat, thereby changing the battery temperature according to the first-order temperature ODE given by

$$
\begin{equation*}
C_{\mathrm{b}} \cdot \frac{\mathrm{~d} \vartheta_{\mathrm{b}}}{\mathrm{~d} t}(s)=P_{\mathrm{b}, 1}(s)-P_{\mathrm{b}, \mathrm{c}}(s) \tag{46}
\end{equation*}
$$

where $C_{\mathrm{b}}$ is the total lumped thermal capacity of the battery, $\vartheta_{\mathrm{b}}(s)$ is the temperature of the battery and $P_{\mathrm{b}, \mathrm{c}}(s) \geq 0$ represents the power outflow from the battery cells to the cooling liquid. Since we consider a battery cooling circuit where the coolant temperature can be actively controlled, the cooling power is free within the bounds defined as

$$
\begin{equation*}
0 \leq P_{\mathrm{b}, \mathrm{c}}(s) \leq \frac{\vartheta_{\mathrm{b}}(s)-\vartheta_{\mathrm{b}, \mathrm{c}}}{\kappa_{\mathrm{b}}}, \tag{47}
\end{equation*}
$$

where $\vartheta_{\mathrm{b}, \mathrm{c}}$ represents the lowest achievable temperature of the cooling liquid and $\kappa_{\mathrm{b}}$ is the thermal resistance between the battery cells and the cooling liquid, where we again assume both parameters to be constant. Rewriting (46) to space domain results in

$$
\begin{equation*}
C_{\mathrm{b}} \cdot \frac{\mathrm{~d} \vartheta_{\mathrm{b}}}{\mathrm{~d} s}(s)=F_{\mathrm{i}}(s)-F_{\mathrm{b}}(s)-F_{\mathrm{b}, \mathrm{c}}(s), \tag{48}
\end{equation*}
$$

where $F_{\mathrm{b}, \mathrm{c}}(s) \geq 0$ is the force-equivalence of the battery cooling power. Note that we explicitly use the difference between the internal battery force and the battery force at the terminals to include the additional charging losses. Similarly as with the EM cooling, we approximate the battery temperature as

$$
\begin{equation*}
\bar{\vartheta}_{\mathrm{b}}(s)=x_{\mathrm{b}, \vartheta}^{\top}(s) Q_{\mathrm{b}, \vartheta} x_{\mathrm{b}, \vartheta}(s)+\vartheta_{\mathrm{b}, 0} \tag{49}
\end{equation*}
$$

where $x_{\mathrm{b}, \vartheta}(s)=\left[1 \vartheta_{\mathrm{b}}(s)\right]^{\top}, \vartheta_{\mathrm{b}, 0}$ is an offset required to obtain positive values and $Q_{\mathrm{b}, \vartheta} \in \mathbb{S}_{-}^{2}$ is a negative semidefinite matrix of coefficients, obtained through semi-definite programming. The semi-definite fit of the battery temperature is similar to Fig. 4, except that the normalized RMSE is reduced to $0.034 \%$ due to the smaller temperature window.


Fig. 7. Comparison between the thermal model and three data sets, of which two represent a dynamic discharge (blue and green) and one represents a constant current charge profile followed by a cooldown period (red). The average normalized RMSE is $0.87 \%$ w.r.t. the maximum temperature of each data set.

Translating (47) to forces and substituting the offset of (49) results in

$$
\begin{equation*}
F_{\mathrm{b}, \mathrm{c}}(s)=\frac{\bar{F}_{\mathrm{b}, \mathrm{c}}(s)+\left(\vartheta_{\mathrm{b}, 0}-\vartheta_{\mathrm{b}, \mathrm{c}}\right) \cdot \frac{\mathrm{d} t}{\mathrm{~d} s}(s)}{\kappa_{\mathrm{b}}} \tag{50}
\end{equation*}
$$

where $\bar{F}_{\mathrm{b}, \mathrm{c}}(s)$ is an intermediate variable used to obtain a convex formulation (see Appendix) through

$$
\begin{equation*}
\bar{F}_{\mathrm{b}, \mathrm{c}}(s) \cdot v(s) \leq x_{\mathrm{b}, \vartheta}^{\top}(s) Q_{\mathrm{b}, \vartheta} x_{\mathrm{b}, \vartheta}(s) . \tag{51}
\end{equation*}
$$

As the coolant temperature cannot exceed the battery temperature, we define

$$
\begin{equation*}
F_{\mathrm{b}, \mathrm{c}}(s) \geq 0 \tag{52}
\end{equation*}
$$

To obtain the thermal parameters of the battery, we apply the convex loss- and cooling model to a combination of data sets recorded from vehicle telemetry. Fig. 7 shows a comparison between the battery temperature from various data sets and the thermal model.

To ensure safe operation of the battery, we define an upper bound on the temperature through

$$
\begin{equation*}
\vartheta_{\mathrm{b}}(s) \leq \vartheta_{\mathrm{b}, \max } \tag{53}
\end{equation*}
$$

where $\vartheta_{\mathrm{b}, \text { max }}$ is the maximum temperature of the battery. Due to the relatively long charge time compared to the refueling time of a conventional race car, it is essential to minimize the charge time. Therefore, we assume that the battery temperature reaches the upper limit at the end of charging, since the temperature is the main limitation. Thus we enforce the initial battery temperature to be at the upper bound through

$$
\begin{equation*}
\vartheta_{\mathrm{b}}(0)=\vartheta_{\mathrm{b}, \max } . \tag{54}
\end{equation*}
$$

Lastly, we specify a terminal value for the battery temperature as

$$
\begin{equation*}
\vartheta_{\mathrm{b}}\left(S_{\mathrm{stint}}\right) \leq \vartheta_{\mathrm{b}, \mathrm{~N}}, \tag{55}
\end{equation*}
$$

where $\vartheta_{\mathrm{b}, \mathrm{N}}$ is the terminal value for the temperature and is calculated during pre-processing using a lookup table having the charge time as an input.

The energy consumption of the battery is modeled as

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} s} E_{\mathrm{b}}(s)=-F_{\mathrm{i}}(s), \tag{56}
\end{equation*}
$$

and we constrain the battery energy as

$$
\begin{equation*}
E_{\mathrm{b}, \min } \leq E_{\mathrm{b}}(s) \leq E_{\mathrm{b}, \max } \tag{57}
\end{equation*}
$$

where $E_{\mathrm{bat}, \text { min }}$ and $E_{\mathrm{bat}, \text { max }}$ correspond to the battery energy at the lower and upper State of Energy (SoE) bound, respectively. Since the voltage of the battery increases with the battery energy, it is optimal to operate the battery at higher energy levels. Therefore, we set the initial battery energy to the upper bound and constrain the terminal battery energy according to the amount of energy charged after driving as

$$
\begin{align*}
E_{\mathrm{b}}(0) & =E_{\mathrm{b}, 0}  \tag{58}\\
E_{\mathrm{b}}\left(S_{\mathrm{stint}}\right) & \geq E_{\mathrm{b}, 0}-E_{\mathrm{b}, \text { charge }} \tag{59}
\end{align*}
$$

where $E_{\mathrm{b}, 0}$ is the initial battery energy and $E_{\mathrm{b}, \text { charge }}$ is the energy the battery receives during charging. In this way, the battery is guaranteed to be charged back to the upper energy bound in energy-limited scenarios. To calculate the battery energy during charging, we leverage a lookup table with input charge time $t_{\text {charge }}$ and output $E_{\mathrm{b}, \text { charge }}$ for a given charging current profile during pre-processing.

## E. Low-level Optimization Problem

This section presents the minimum-stint-time control problem of the electric race car. Given a predefined stint length and charge time we formulate the control problem using the state variables $x=\left(E_{\mathrm{kin}}, E_{\mathrm{b}}, \vartheta_{\mathrm{b}}, \vartheta_{\mathrm{m}}\right)$ and the control variables $u=\left(F_{\mathrm{m}}, F_{\text {brake, } \mathrm{F}}, F_{\text {brake,R }}\right)$ as follows:

Problem 1 (Minimum-stint-time Control Strategy). The minimum-stint-time control strategies are the solution of

$$
\begin{aligned}
& \min \int_{0}^{S_{\text {stint }}} \frac{\mathrm{d} t}{\mathrm{~d} s}(s) \mathrm{d} s \\
& \text { s.t. }(2)-(19),(22),(25),(27)-(31),(33) \\
& \quad(35),(39)-(45),(48),(50)-(59)
\end{aligned}
$$

Since the feasible domain and the cost function are convex, the low-level control problem is fully convex and therefore we can compute a globally optimal solution with standard nonlinear programming methods.

## III. High-level Race Optimization

In this section, we present the high-level maximum-racedistance control problem. First, we formulate the maximum-race-distance control problem that optimizes the stint length and charge time for a pre-defined number of pit stops. Second, we model the minimum stint time by leveraging the low-level control problem and optimizing for various combinations of stint length and initial battery energy. Finally, we extend the maximum-race-distance control problem to allow joint optimization of the stint length, charge time, and number of pit stops.

## A. Mixed-integer Control Problem

We define the high-level control problem for a pre-defined number of pit stops in stint domain, so that we have a fixed and finite optimization horizon. Here, each index in the optimization variables represents a stint. The goal is then to maximize the driven distance as the sum of all completed laps during the stints as

$$
\begin{equation*}
\max S_{\mathrm{race}}=\max \sum_{k=0}^{n_{\text {stops }}} S_{\mathrm{lap}} \cdot N_{\mathrm{laps}}(k), \tag{60}
\end{equation*}
$$

where $S_{\text {race }}$ is the total race distance, $n_{\text {stops }}$ is the predefined number of pit stops, $N_{\text {laps }}(k) \in \mathbb{N}, \forall k \in$ $\left[0, \ldots, n_{\text {stops }}-1\right]$ is the stint length and $\mathbb{N}$ the set of natural numbers, and $S_{\text {lap }}$ is the length of one lap. Since the vehicle starts and stops at the pit box, the stint length should be an integer number of laps. As it is unlikely that the vehicle is exactly at the finish line when the race time limit is reached, we allow the final stint length to be a non-integer number of laps. This way, we have $n_{\text {stops }}+1$ stints for $n_{\text {stops }}$ pit stops and thus we have $n_{\text {stops }}$ integer stint lengths and one final non-integer stint length.

The race can be divided into the car driving a stint and the car recharging the battery during pit-stops. Given the total race time $t_{\text {race }}$, we can link it to the time to complete the stint $t_{\text {stint }}(k) \geq 0$ and the time spent charging $t_{\text {charge }}(k) \geq 0$ as

$$
\begin{equation*}
t_{\mathrm{race}}=\sum_{k=0}^{n_{\text {stops }}} t_{\text {stint }}(k)+\sum_{k=0}^{n_{\text {stops }}-1} t_{\text {charge }}(k) \tag{61}
\end{equation*}
$$

We then decompose the total race into blocks consisting of the vehicle first driving a stint followed by a pit stop in which the battery is charged. Assuming that a stint is always energylimited, the charge time uniquely defines the terminal battery energy for the prior stint and is not influenced by other stints. Furthermore, we assume that the battery is thermallylimited during charging, which allows us to pre-calculate the maximum terminal battery temperature for the prior stint through backwards integration of the battery temperature dynamics during charging. Thereby, we uniquely base the terminal battery temperature on the charge time, without being influenced by other stints. Lastly, we assume that the EM temperature reaches the upper limit at the end of driving and assume that the charge times across consecutive stints remains constant. This allows us to calculate the EM temperature at the beginning of the stint by integrating the EM temperature dynamics during charging. This way, we uniquely define the initial EM temperature on the charge time.

To ensure that the battery is not overcharged, we apply an upper bound on the charge time through

$$
\begin{equation*}
t_{\text {charge }}(k) \leq t_{\text {charge, } \max }, \tag{62}
\end{equation*}
$$

where $t_{\text {charge, max }}$ is the maximum charge time corresponding to charging the battery from the lower to the upper energy level. Finally, the time to complete the stint is obtained by solving the low-level control problem, which we explain in the next section.

## B. Stint Time Model

In this section, we derive a method for modeling the stint time as a function of the stint length and charge time during the pit stop prior to the stint. We solve the low-level control Problem 1 for a combination of stint lengths and charge times to obtain the respective achievable minimum stint time. This way, we can create the lookup table with stint time as a function of stint length and charge time shown in Fig. 8. Thereby, the charge time and terminal battery energy are linked through a pre-defined charging profile, cf. Section IID, whereas the terminal battery temperature and initial EM temperature are calculated during pre-processing based on the charge time solely, cf. Section II-B, II-D. As the stint time increases for larger stint lengths and shorter charge times, similar to the EM loss fit in Section II-B above, we approximate the low-level optimization results via the continuous function

$$
\begin{equation*}
t_{\mathrm{stint}}(k)=x_{\mathrm{s}}^{\top}(k) Q_{\mathrm{s}} x_{\mathrm{s}}(k) \tag{63}
\end{equation*}
$$

where $x_{\mathrm{s}}(k)=\left[\frac{1}{\sqrt{t_{\text {charge }}(k)}} \sqrt{t_{\text {charge }}(k)} \frac{N_{\text {laps }}(k)}{\sqrt{t_{\text {charge }}(k)}}\right]^{\top}$ and $Q_{\mathrm{s}} \in \mathbb{S}_{+}^{3}$ is a symmetric positive semi-definite matrix of coefficients. The result of the fit is shown in Fig. 8. For a convex implementation, we relax and rewrite (63) to

$$
\begin{equation*}
t_{\mathrm{stint}}(k) \cdot t_{\text {charge }}(k) \geq y_{\mathrm{s}}(k)^{\top} Q_{\mathrm{s}} y_{\mathrm{s}}(k), \tag{64}
\end{equation*}
$$

where $y_{\mathrm{s}}(k)=\left[1 t_{\text {charge }}(k) N_{\text {laps }}(k)\right]^{\top}$, and convert this relaxation to a conic constraint [29] as

$$
t_{\text {stint }}(k)+t_{\text {charge }}(k) \geq\left\|\begin{array}{c}
2 \cdot z_{\mathrm{s}}(k)  \tag{65}\\
t_{\text {stint }}(k)-t_{\text {charge }}(k)
\end{array}\right\|_{2},
$$

where $z_{\mathrm{s}}=L_{\mathrm{s}} y_{\mathrm{s}}(k)$ with $L_{\mathrm{s}}$ being the Cholesky factorization of $Q_{\mathrm{s}}$ [29]. Since it is optimal to minimize stint time, this constraint will hold with equality at the optimum.

The final stint of the race is not followed by a pit stop in which the battery is charged. Therefore, the battery can be fully depleted and there is no margin needed in the battery temperature. Therefore, we separately model the final stint by solving the low-level control Problem 1 for a range of stint lengths, with a fixed charge time $t_{\text {charge }}\left(n_{\text {stops }}\right)=$ $t_{\text {charge, max }}$ and terminal battery temperature $\vartheta_{\mathrm{b}}\left(S_{\text {stint }}\right)=$ $\vartheta_{\mathrm{b}, \max }$. With the charge time being fixed, we can then model the final stint time by a quadratic function with the stint length as

$$
\begin{equation*}
t_{\text {stint }}\left(n_{\text {stops }}\right) \geq D_{\mathrm{s}, \mathrm{f}}^{\top} x_{\mathrm{s}, \mathrm{f}} \tag{66}
\end{equation*}
$$

where $D_{\mathrm{s}, \mathrm{f}}$ is a vector of coefficients and $x_{\mathrm{s}, \mathrm{f}}=$ $\left[N_{\text {laps }}^{2}\left(n_{\text {stops }}\right) N_{\text {laps }}\left(n_{\text {stops }}\right) 1\right]^{\top}$. Fig. 9 shows the quadratic fit of the final stint time model.

## C. Optimal Pit Stop Strategy

In the previous sections, we introduced the objective and constraints for the high-level control problem when optimizing the race strategy for a pre-defined number of pit stops. In this section, we apply some modifications in order to jointly optimize the stint lengths, charge times and number of pit stops, thereby removing the need to search over a large space of pre-defined number of pit stops.


Fig. 8. Fit of optimization results for a combination of stint lengths and charge times. The normalized RMSE of the fit is $2.2 \%$ w.r.t. the maximum stint time.


Fig. 9. Fit of the final stint times for a range of stint lengths. The normalized RMSE of the fit is $0.58 \%$ w.r.t. the maximum stint time.

We define a binary variable $b_{\text {pit }}(k)$ that indicates whether pit stop and stint $k$ is taken or skipped as

$$
b_{\mathrm{pit}}(k)= \begin{cases}0, & \text { if stint and stop skipped }  \tag{67}\\ 1, & \text { if stint and stop taken }\end{cases}
$$

and include it in (63) via the big-M formulation [30]

$$
\begin{equation*}
t_{\mathrm{stint}}(k) \geq x_{\mathrm{s}}(k)^{\top} Q_{\mathrm{s}} x_{\mathrm{s}}(k)-M \cdot\left(1-b_{\mathrm{pit}}(k)\right) \tag{68}
\end{equation*}
$$

where $M \gg t_{\text {stint, max }}$. This way, we obtain the original constraint if $b_{\text {pit }}(k)=1$ and we obtain a negative lower bound when $b_{\text {pit }}(k)=0$. By defining

$$
\begin{align*}
t_{\text {stint }}(k) & \geq 0,  \tag{69}\\
t_{\text {charge }}(k) & \geq 0, \tag{70}
\end{align*}
$$

the $k$-th stint time and charge time will be pushed to zero, hence skipping the stint. We convert (68) to a cone as

$$
\begin{gather*}
M \cdot\left(1-b_{\mathrm{pit}}(k)\right)+t_{\text {stint }}(k)+t_{\text {charge }}(k) \geq \\
2 \cdot z_{\mathrm{s}}(k)  \tag{71}\\
\left\|\cdot\left(1-b_{\text {pit }}(k)\right)+t_{\text {stint }}(k)-t_{\text {charge }}(k)\right\|_{2} .
\end{gather*}
$$

Hence, whenever a stint is skipped, the corresponding stint time and charge time will be zero if an optimal solution is obtained. To prevent the stint length from diverging to infinity whenever the stint is actually skipped, i.e., $b_{\text {pit }}(k)=$ 0 , we define an upper bound on stint length as

$$
\begin{equation*}
N_{\text {laps }}(k) \leq N_{\text {laps }, \text { max }} \cdot b_{\text {pit }}(k), \tag{72}
\end{equation*}
$$

where $N_{\text {laps, max }}$ is the maximum stint length that was used to obtain the lookup table. This will ensure $N_{\text {laps }}(k)=0$ whenever $b_{\text {pit }}(k)=0$. Since the final stint is not constrained
by a lower terminal battery temperature and can deplete the battery without spending time on charging afterwards, the final stint has to be part of the optimal race strategy. We ensure this by writing

$$
\begin{equation*}
b_{\mathrm{pit}}(k+1) \geq b_{\mathrm{pit}}(k), \quad \forall k \in\left[0, n_{\text {stops }}-1\right] \tag{73}
\end{equation*}
$$

Suppose we set the length of the variables to $N$ and that $x$ stints and stops are skipped. Then the first $x$ entries in $b_{\text {pit }}$ will be zero and the last $N-x$ entries will be one. This way, the final stint is always taken and the search space for the solver is reduced.

## D. High-level Optimization Problem

This section presents the maximum-race-distance control problem of the electric race car. Given a predefined race time we formulate the control problem using the control variables ( $t_{\text {charge }}, N_{\text {laps }}, b_{\text {pit }}$ ) as follows:

Problem 2 (Maximum-race-distance Strategies). The maximum-race-distance strategies are the solution of

$$
\begin{aligned}
& \max \sum_{k=0}^{n_{\text {stops }}} S_{\text {lap }} \cdot N_{\text {laps }}(k) \\
& \text { s.t. }(61),(62),(66),(69)-(73)
\end{aligned}
$$

Since Problem 2 can be solved with mixed-integer secondorder conic programming solvers, we can guarantee global optimality upon convergence [31], [32].

## IV. Discussion

A few comments are in order. First, we assume that endurance racing tires do not degrade significantly and can be changed every stint due to the long pit stop time. Yet the high-level control problem can be readily extended to capture these dynamics if the lookup table is devised accounting for tire degradation. Second, we assume that the time gained from starting the race from the grid compared to the pit lane is negligible on a full endurance race. Thus we do not separately optimize the first stint. Third, when the battery temperature is not an active constraint, the battery cooling relaxes in order to maximize the battery efficiency. Yet this can be interpreted as the battery coolant temperature being controlled by the refrigerant cooling system, which allows reduced cooling power, assuming that the system can cope with the requested coolant temperature changes. Fourth, in scenarios where the battery or EM temperature is very limiting and the lower battery energy limit is not reached, it can occur that (2) relaxes due to the positive contribution in (27), (39) and (50). Yet, these situations will not be part of the optimal race strategy, since charging the battery to such an extent that the available energy during driving cannot be used is sub-optimal. In fact, these situations were not found to be part of the optimal strategies in this study, as will be shown in the next section. If the solution where the lethargy relaxes is not part of the optimal race strategy, it is guaranteed that the physically correct solution will not be part of the optimal strategy either, since it is always worse or equal. Finally, we assume a fixed value for the terminal EM temperature when pre-calculating the bound on the initial


Fig. 10. Velocity, EM power and battery SoE trajectories per lap for an 11 lap stint. The battery energy is an active constraint, thus the stint is energylimited. The EM power shows a gradual decrease at high velocities, thus indicating energy management.

EM temperature. However, it can occur that this value is not reached, e.g., in scenarios where the EM temperature is not an active constraint. In these situations, the temperature trajectories might not reflect reality, yet this does not affect the resulting solution, as there are no other states that depend on the EM temperature.

## V. Results

This section presents numerical results for both the lowand high-level control problem. We base our use case on the rear-wheel driven electric endurance race car of InMotion [33], shown in Fig. 1, performing an 11 lap stint at the Zandvoort circuit for the low-level control problem and a 6 h race at the same circuit for the high-level control problem. First, we discuss the numerical solutions for both control problems. Second, we validate the high-level control problem by comparing the optimal race strategy against fixed-pit-stopnumber strategies and compare the results to the expected optimal combinations of stint length and charge time.

For the discretization of the model, we apply the trapezoidal method with a fixed step-size of $\Delta s=4 \mathrm{~m}$. We parse the low-level control problem with CasADi [34] and solve it using IPOPT [35] combined with the MA57 linear solver [36], whilst we parse the high-level control problem with YALMIP [37] and solve it using MOSEK [38]. We perform the numerical optimization on an Intel Core i74710 MQ 2.5 GHz processor and 8 GB of RAM. Thereby, the computation time for solving the low-level problem was about 0.68 s of parsing and 61 s of solving, whereas the highlevel problem needed 0.15 s of parsing and 8.1 s of solving.

## A. Low-level Optimization

In this section, we compute the optimal trajectories for a stint of 11 laps around the Zandvoort circuit. We set the terminal battery capacity to the energy level corresponding to a 5 min charge time using constant current charging, which means that the battery is partially charged. The total stint


Fig. 11. Velocity, power loss, battery SoE, battery temperature and EM temperature trajectories for an 11 lap stint. The battery energy is an active constraint, thus the stint is energy-limited and energy management is needed. Furthermore, the maximum EM temperature is reached multiple times throughout the stint. Therefore, the velocity profile decreases when the maximum EM temperature is reached.


Fig. 12. The friction circles per axle showing the normalized longitudinal and lateral forces. The vehicle is rear-wheel driven and braking is done mostly using the EM. Therefore, the front axle (left figure) mostly generates lateral forces, whereas the rear axle (right figure) shows more combined forces.
time is about 1127 s with an average flying lap time of 101 s (not counting the first and last lap).

The velocity profile together with the EM power and SoE per lap is shown in Fig. 10. Furthermore, the total stint velocity profile together with the powertrain losses, SoE, battery temperature and EM temperature is shown in Fig. 11. First, we observe that the velocity profiles of consecutive free-flow laps are slightly different, which is due to the EM temperature being an active constraint. Since the EM temperature starts below the limit, the velocity in the first laps is the highest and decreases as the EM


Fig. 13. Evolution of the completed laps as a function of time for the optimal strategy (black) and the strategies optimized for a fixed number of pit stops. Jointly optimizing the number of stints can significantly outperform other strategies by multiple laps.
temperature reaches the upper limit. In this scenario, the battery temperature is not an active constraint, since the battery temperature is kept at the maximum level during the first half of the stint by reducing the cooling power, thereby increasing the battery efficiency. In the second half of the stint, the battery temperature decreases gradually by applying maximum cooling to reach the terminal value. Second, the EM power decreases gradually before the vehicle reaches a corner and regenerative braking is applied. However, both the power during traction as well as the regenerative braking power decrease when the EM temperature limit is reached. In general, the velocity profile shows smooth behavior, which is typical for energy-limited scenarios, since regenerative braking is used to reduce the velocity before cornering instead of the mechanical brakes. Third, we observe that the battery energy exceeds the terminal value before the end of the stint. Since the battery is partially charged in this scenario, the absolute lower energy limit is not reached. With the use of regenerative braking, the battery energy then reaches the terminal value exactly at the end of the stint, indicating an energy-limited scenario. Finally, Fig. 12 shows the normalized lateral and longitudinal per axle, along with the maximum grip limit defined by the friction coefficient. Since the vehicle in this study is rear-wheel driven, the front axle only provides negative forces in the longitudinal direction. It is noticeable that the data points do not show the typical pattern for the front axle that is expected in racing scenarios [39]. This is because the mechanical brakes are only used during pit entry at the end of the stint and because the vehicle applies regenerative braking to slow down for cornering. Moreover, both the front and rear axles are operated at the lateral grip limit in corners, indicating that the cornering velocity is maximized.

## B. High-level Optimization

This section presents the optimal race strategy in terms of number of pit stops, stint length and charge time, and we


Fig. 14. Optimal race strategy (black) in terms of stint length, charge time and stint time with $t_{\text {charge, } \max }=7.5 \mathrm{~min}$. For comparison, we show other optimal fixed pit stops strategies together with the relaxed solution in gray. Stint length, charge- and stint time are related and the optimal integer solution minimizes the differences to the relaxed solution.
compare it against the strategies optimized for a fixed number of pit stops. We select a 6 h race, yet longer races can be solved as well with our approach, considering the very low computational times needed by our high-level framework to converge. To link the terminal battery energy $E_{\mathrm{b}}\left(S_{\text {stint }}\right)$ to the charge time $t_{\text {charge }}$, we apply constant current charging. Furthermore, we calculate the terminal battery temperature $\vartheta_{\mathrm{b}, \mathrm{N}}$ using the losses that correspond to the constant current profile and obtain the initial EM temperature $\vartheta_{\text {m,init }}$ by applying maximum cooling, for the given charge time.

Fig. 13 shows the evolution of the completed laps as a function of time for various fixed pit stop strategies. We observe that the optimal strategy of 15 stops results in the largest amount of completed laps, thereby confirming that it is indeed optimal in terms of number of pit stops. The difference in covered race length between the optimal and fixed-pit-stop-number strategies can exceed multiple laps and hence significantly affect the final race outcome in terms of finishing position, highlighting the importance of jointly optimizing the number of pit stops.

Fig. 14 shows the individual stints in terms of length and charge time, together with the relaxed non-integer solution. We can conclude that a constant stint length over the race is optimal, since all stints in the relaxed solutions are equal, with the only exception being the last stints. In this use case, the optimal integer solution consists of the stint lengths that minimize the difference to the relaxed solution, namely, of a stint length between 10 and 11 laps together with a charge time of about 4.2 min and 15 pit stops in total. For the vehicle considered in this study, the 8 stop strategy is the fastest strategy that involves fully charging the battery. However, this strategy is 0.6 laps behind on the optimal race strategy, thereby showing that fully charging the battery can be suboptimal. This is explained by the constant current charging,


Fig. 15. The average stint velocity for a combination of stint lengths and charge times together with the optimal combinations and actual numerical solutions. The optimal combinations of stint length and charge time show a clear (linear) correlation, to which the numerical solutions are aligned.
which effectively reduces the average charging power for longer charge times due to the battery voltage decreasing for lower energy levels. With a different charging profile, such as constant power charging, a different optimal race strategy will be obtained. However, it is beyond the scope of this thesis to optimize the charging profile. Moreover, reducing the charge time results in shorter stints, thereby increasing the number of pit stops needed, as indicated by the results. This then results in more time lost in the pit lane due to the speed limit. Thus there is a clear trade-off being made in determining the optimal charge time. When we also consider thermally-limited scenarios, this trade-off becomes even more complex. In these cases, a shorter charge time is expected to be in favor, since this increases the terminal battery temperature. Although a longer charge time reduces the initial EM temperature, it was observed that this has a relatively small impact on the stint time. In fact, it was observed that in longer stints the EM was operated relatively more at the thermal limit compared to shorter stints, thereby disfavoring longer charge times. Lastly, we observe that there is a considerable decrease in race distance when an increasing stint length cannot be compensated by an increase in charge time, as illustrated by the difference between the 7 and 8 stop strategies. From the aforementioned observations, we conclude that the stint length, stint time and charge time are closely related in case of an optimal solution.

## C. Validation

In this section, we validate the correctness of the model by showing that the lethargy constraint holds with equality for various stints that were presented in the previous section. Furthermore, we validate the convex models by implementing the optimal inputs into a non-linear simulator and compare the drift in battery energy. Finally, we validate the numerical combinations of stint length and charge time for the various strategies. For the latter, we calculate the


Fig. 16. Battery energy calculated through the non-linear and convex models, using the optimized inputs for the globally optimal stint of 10 laps and 4.2 min charging.
average stint velocity $\bar{v}_{\text {stint }}(k)$ for every strategy as

$$
\begin{equation*}
\bar{v}_{\text {stint }}(k)=\frac{S_{\text {stint }}(k)}{t_{\text {charge }}(k)+t_{\text {stint }}(k)}, \quad \forall k \in\left[0, n_{\text {stops }}-1\right] \tag{74}
\end{equation*}
$$

The globally optimal stint should maximize the average stint velocity, since that maximizes the driven distance per unit of time. Fig. 15 shows the average stint velocity for all possible combinations of stint length and charge time together with the theoretical optimal charge times that maximize the average stint velocity for a given stint length, to which we refer as the optimal combinations. These optimal combinations show an almost linear relation between charge time and stint length until the maximum charge time is reached. The most noticeable exception is the single-lap stint, which shows a relatively higher optimal charge time. This is because a single-lap stint does not include a flying-lap, where the vehicle starts the lap with a high initial velocity. Thus, the vehicle requires more energy in the first lap to accelerate towards the optimal velocity, which explains the longer charge time for a single-lap stint. The globally optimal stint consists of 10 laps and 4.2 min charging, which is the exact same combination that we obtained as the optimal strategy in the previous section. Furthermore, we observe that the average stint velocity decreases in sensitivity around the optimal combinations for increasing stint length and charge time. When the maximum charge time is reached, the average stint velocity diminishes considerably for increasing stint lengths. Thereby, increasing the stint length beyond 18 laps quickly becomes less favorable, since it cannot be compensated by an increase in charge time. This explains why the 7 stop strategy is significantly worse than the others. Finally, we note that the numerical solutions are in line with the theoretically optimal combinations. The outliers not aligning with the optimal combinations, e.g., at 16.4 laps and 7.5 min charging, correspond to the last stints, for which the charge time is not part of the race and thus the calculation of the stint velocity in (74) is not valid.

To validate the convex loss models for the EM and battery, we calculate the battery energy trajectory by applying the optimal input trajectories to non-linear models and compare the result to the trajectory obtained from the optimization. Fig 16 shows both battery energy trajectories for the globally optimal stint of 10 laps with 4.2 min charging. From this figure, we observe small deviations, with a total drift of $-1.53 \%$ with respect to the non-linear models, thereby indicating that


Fig. 17. Validation of the lethargy constraint for the mode of the stints obtained from the high-level optimization.
the convex models accurately capture the dynamics of the powertrain components.
Finally, we verify that all relaxed constraints hold with equality. As noted previously, it can occur that the lethargy constraint does not hold with equality in thermally-limited scenarios. Therefore, we explicitly show this constraint for the most frequently used stints per pit stops strategy that were obtained in the previous section. Fig 17 shows that all data points align with the constraint, thereby indicating that the lethargy constraint holds with equality for all optimal stints.

## VI. Conclusion

In this thesis, we devised a bi-level optimization framework to efficiently solve the maximum-distance endurance race strategy problem for a fully electric race car. In order to tackle the large problem size stemming from the length of an endurance race, we decomposed the problem into separate stints, which we solved by extending a minimum-lap-time convex optimization framework that can rapidly deliver the globally optimal solution to capture multiple laps and include more accurate force-based models to account for thermal limitations, whilst including the vehicle dynamics directly in the optimization using a bicycle model. This way, we were able to compute the optimal number of pit stops, the charging time per stop and the individual stint lengths via mixed-integer second-order conic programming with global optimality guarantees. Our bi-level framework could solve the problem of a 6 h race around the Zandvoort circuit with low computation times below 10 s for the high-level framework. Our results showed that, from a stint perspective, there is a clear correlation between optimal stint length and charge time, which corresponds to the maximization of the average stint velocity. Moreover, the results showed that the optimal race strategy can involve partially charging the battery, depending on the charging profile. Finally, we highlighted the importance of optimizing both levels and that, compared to the strategies optimized for a pre-defined number of pit stops, jointly optimizing the number of pit stops can significantly increase the total distance driven by multiple laps, hence considerably improving the achievable race outcome.

This work opens the field for the following possible extensions: First, the model could be extended to account for the temperature dynamics of the cooling liquids, since they can limit the available cooling power. Second, we are interested in the impact of tire degradation on the achievable
stint time and the resulting race strategies. Finally, we would like to derive a method to allow drivers to provide the optimal inputs to the vehicle and validate the results on a real vehicle.

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## Appendix

In Section II-E, we frequently used the following relaxations

$$
\begin{equation*}
x \cdot y \geq z^{\top} z \tag{75}
\end{equation*}
$$

which defines a convex set. To prove convexity, we can rewrite this constraint to a second-order conic constraint as

$$
x+y \geq\left\|\begin{array}{c}
2 \cdot z  \tag{76}\\
x-y
\end{array}\right\|_{2}
$$

which can be solved with global optimality guarantees [29]. Since (75) is mathematically equivalent to (76), both optimization problems will converge to the same KKT points, thereby guaranteeing global optimality.

As an alternative to second-order conic programming, we can write (75) as a semi-definite constraint through

$$
\left[\begin{array}{ll}
x & z  \tag{77}\\
z & y
\end{array}\right] \succeq 0
$$

which can be solved to global optimality with semi-definite programming solvers.

Another type of constraint that was used in Section II-E for the friction circles is

$$
\begin{equation*}
x^{2}+y^{2} \leq z^{2} . \tag{78}
\end{equation*}
$$

This constraint can be directly translated to a second-order conic constraint as

$$
\left\|\begin{array}{l}
x  \tag{79}\\
y
\end{array}\right\|_{2} \leq z
$$

which again can be solved to global optimality [29], thereby proving that (78) defines a convex set.


[^0]:    i See: https://www.tue.nl/en/our-university/about-the-university/organization/integrity/scientific-integrity/
    The Netherlands Code of Conduct for Scientific Integrity, endorsed by 6 umbrella organizations, including the VSNU, can be found here also. More information about scientific integrity is published on the websites of TU/e and VSNU

