# Obstacle induced particle jamming in exclusion dynamics

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## Particles in a strip

- model
- stationary state
- residence time

## Strip without obstacles

- effect of lateral displacement
- mean field or macroscopic limit
- analogy with a not symmetric Random Walk
- results

## Strip with obstacles

- mean field stationary state
- results on residence time

# The problem

Propose a basic model to study the effect of obstacles in a strip in which particles are moving possibly in a preferred direction.

Question we address: effect of the obstacles on the *residence time*, that is to say, the typical time needed by the particles to walk along the whole lane?

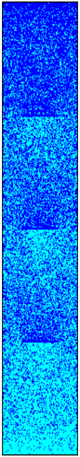
In collaboration with A. Muntean, O. Krehel, R. van Santen, A. Sengar,

- “Residence time estimates for asymmetric simple exclusion dynamics on strips”, Physica A **4422**, 436–457, 2016

In collaboration with A. Muntean, O. Krehel, R. van Santen,

- “A lattice model of reduced jamming by barrier”, Physical Review E **94**, 042115, 2016

# Flux in a strip with obstacles



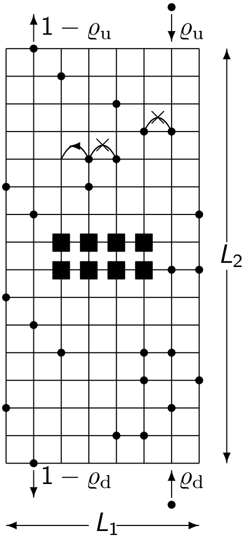
The motion of particles obeys the following rules:

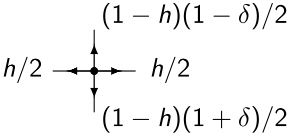
- particle move randomly in any direction on a square lattice (the strip);
- particles enter the strip through the top boundary;
- particles exit the strip through the bottom boundary;
- top and vertical boundaries are reflecting for particles moving inside the strip;
- particle possibly experiences a downward drift.

Possible impediments to motion:

- obstacles in the core of the strip modeled by rectangles with reflecting boundaries;
- particles entering the strip through the bottom boundary.

# Model




- $(y,x) \in \{1,\dots,L_1\} \times \{1,\dots,L_2\}$ lattice site
- simple exclusion in the bulk
- reflecting vertical boundaries
- reservoirs $\varrho_u = 1$ and $\varrho_d \in [0,1)$
- $h \in [0,1)$ probability of horizontal motion
- $\delta \in [0,1]$ vertical drift
- black squares denote obstacles

# Model

At each time step $t$ try to move a number of particles equal to the number of particles $n(t-1)$ in the strip at $t-1$ plus one.

For $n(t-1)+1$ times with probabilities

$$\frac{\varrho_{\mathrm{u}} L_1}{\varrho_{\mathrm{u}} L_1 + \varrho_{\mathrm{d}} L_1 + n(t)}, \frac{\varrho_{\mathrm{d}} L_1}{\varrho_{\mathrm{u}} L_1 + \varrho_{\mathrm{d}} L_1 + n(t)}, \frac{n(t)}{\varrho_{\mathrm{u}} L_1 + \varrho_{\mathrm{d}} L_1 + n(t)}$$

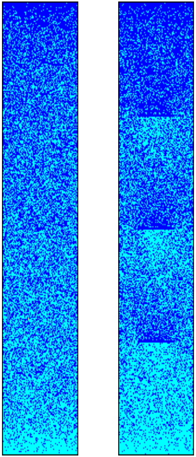
do the following: Top insert, Bottom insert, Bulk move.

Top insert: choose uniformly a site in row 1 and put there a new particle if empty: $n(t) = n(t) + 1$.

Bottom insert: choose uniformly a site in row $L_2$ and put there a new particle if empty: $n(t) = n(t) + 1$.

Bulk move: see the picture.

# Stationary state



Let evolve the system for a sufficiently long time $\bar{t}$ and then for $t$ large the sum

$$\varrho(y,x) = \frac{1}{t-\bar{t}} \sum_{t'=\bar{t}}^{t} \eta_{t'}(y,x)$$

becomes constant and provides the *stationary occupation number profile.*

Horizontal average:

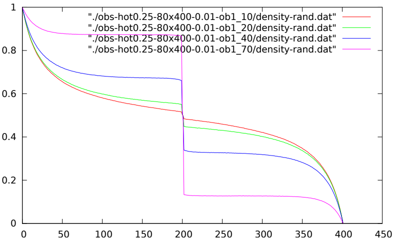
$$\varrho(x) = \frac{1}{L_1} \sum_{x=1}^{L_1} \varrho(y,x)$$

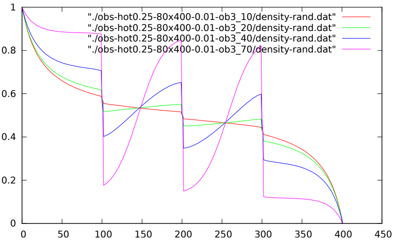
# Stationary state

Parameters: $L_1 = 80$, $L_2 = 400$, $\varrho_{\mathrm{u}} = 1$, $\varrho_{\mathrm{d}} = 0$, $h = 0.5$, and $\delta = 0.04$.

One obstacle on the left and three obstacles on the right.

Obstacle width: red 10, green 20, blue 40, and purple 70.



# Residence time

The main quantity of interest is the *residence time* at stationarity:

$$R = \text{typical time a particle needs to exit the strip at stationarity}$$

Computing the residence time:

- *Monte Carlo* estimate: we shall run long simulations and average *at stationarity* the time needed by each particle which entered the strip through the top boundary to exit through the bottom boundary;
- we shall develop two analytic arguments to estimate the residence time which will be called *Mean Field* (MF) and *Birth–and–Death* (BD) estimates.

# Strip without obstacles

# Effect of lateral displacement

In absence of obstacles we expected a trivial effect due to the two–dimensionality of the problem (positive $h$):

- consider a phenomenon taking time $T$ for the analogous one dimensional problem ($h = 0$);
- in the strip it will take the time $T'$ such that

$$T' = T + hT' \Longrightarrow T' = \frac{T}{1-h}$$

Numerically we find this trivial result in the cases $\delta = 1$ or $\varrho_{\mathrm{d}} = 0$

Numerically we find not trivial results in the cases $\delta = 0$ and $\varrho_{\mathrm{d}} > 0$

- absence of monotony with respect to $h$
- transition between two different analytic results

# Mean Field or Macroscopic Limit

Macroscopic variables under diffusive scaling $\varepsilon \to 0$:

$$y \to \varepsilon y, \;\; x \to \varepsilon x, \;\; t \to \varepsilon^2 t \quad \text{and} \quad \delta \to \varepsilon \delta$$

It is derived a macroscopic equation for the typical occupation number $m_t(y, x)$ for $\varepsilon \to 0$:

$$\frac{\partial m_t}{\partial t} = \frac{1}{2} h \frac{\partial^2 m_t}{\partial y^2} + \frac{1}{2}(1-h)\frac{\partial^2 m_t}{\partial x^2} - \delta(1-h)\frac{\partial}{\partial x}[m_t(1-m_t)]$$

In the one dimensional case this result is rigorous: A. De Masi, E. Presutti, E. Scacciatelli, "The weakly asymmetric simple exclusion process," Ann. Ist. H. Poincaré A **25**, 1–38, 1989.

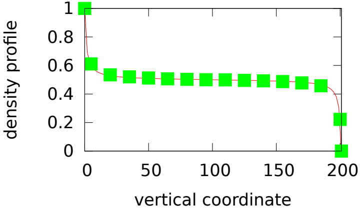
# Mean field stationary profile

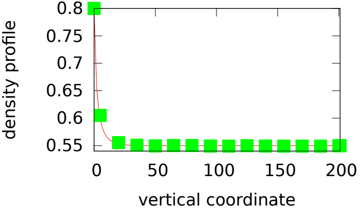
In absence of obstacle: look for stationary solution of the macroscopic equation $\varrho = \varrho(x)$ not depending on the horizontal coordinate $y$.

Get the equation:

$$\frac{1}{2}\frac{d^2}{dx^2}\varrho - \delta\frac{\mathrm{d}}{\mathrm{d}x}\varrho(1-\varrho) = 0 \quad \text{with} \quad \varrho(0) = \varrho_{\mathrm{u}}, \ \varrho(L_2+1) = \varrho_{\mathrm{d}}$$

Parameters: $L_1 = 100$, $L_2 = 200$, $h = 0.5$, and $\delta = 0.8$. Left: $\varrho_{\mathrm{u}} = 1$, $\varrho_{\mathrm{d}} = 0$. Right: $\varrho_{\mathrm{u}} = 0.8$, $\varrho_{\mathrm{d}} = 0.55$.



# Mean Field residence time prediction

The Mean Field equation is a continuity equation for the flux

$$\vec{J_t} = -\frac{1}{2}h\frac{\partial m_t}{\partial y}\vec{e_1} + \Big( -\frac{1}{2}(1-h)\frac{\partial m_t}{\partial x} + \delta(1-h)m_t(1-m_t)\Big)\vec{e_2}$$

At stationarity the density $\varrho$ depends only on $x$, thus the flux reads

$$J = -\frac{1}{2}(1-h)\frac{\partial \varrho}{\partial x}(x) + \delta(1-h)\varrho(x)[1-\varrho(x)] = -\frac{1}{2}(1-h)\varrho'(0)$$

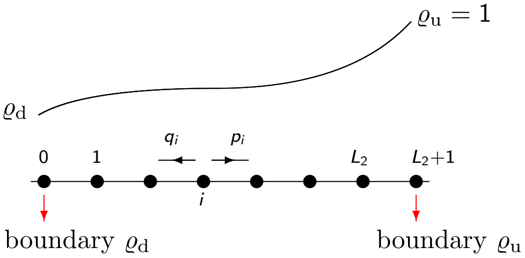
Assume that the typical velocity $v(x)$ of a particle occupying a position with vertical coordinate $x$ is such that

$$\varrho(x)v(x) = J$$

A simple integration gives the residence time

$$R = \int_0^{L_2+1} \frac{\varrho(x)}{J}\,\mathrm{d}x = -\frac{2}{(1-h)\varrho'(0)}\int_0^{L_2+1}\varrho(x)\,\mathrm{d}x$$

# Analogy with a not symmetric Random Walk

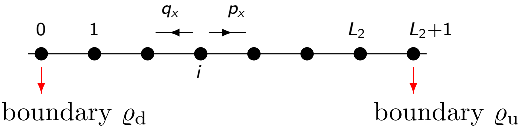



Jump probabilities are chosen as follows:

$$q_i = \frac{1-h}{2}(1+\delta)[1-\varrho(L_2+1-i+1)] \qquad \text{for } i = 1, \ldots, L_2$$

$$p_i = \frac{1-h}{2}(1-\delta)[1-\varrho(L_2+1-i-1)] \qquad \text{for } i = 0, \ldots, L_2 - 1$$

# Analogy with a not symmetric Random Walk




Residence time

$$R \stackrel{?}{=} \mathbb{E}[T]$$

where

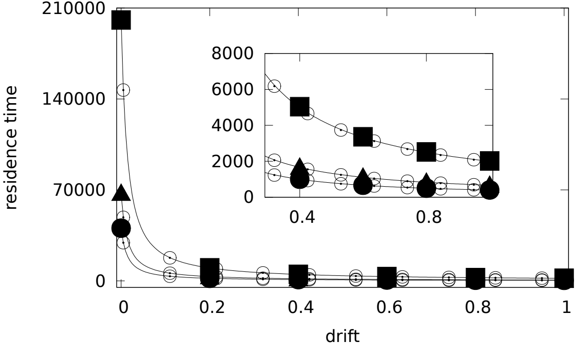
$$T = \text{ first hitting time to } 0 \text{ for a walker started at } L_2$$

It is well known that

$$\mathbb{E}[T] = \frac{1}{q_{L_2}} + \sum_{i=1}^{L_2-1} \frac{1}{q_i} \left(1 + \sum_{j=i+1}^{L_2} \prod_{k=i+1}^{j} \frac{p_{k-1}}{q_k}\right)$$

A general explicit expression cannot be provided, but one can compute the above sum numerical for any choice of the parameters.

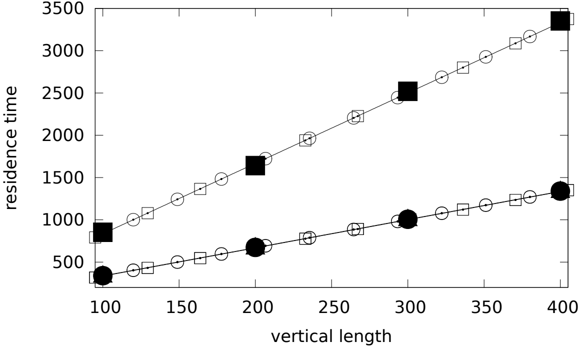
# Results: $\varrho_{\rm d} = 0$




Parameters: $\varrho_{\rm d} = 0$, $L_1 = 100$, and $L_2 = 200$; the symbols •, ▲ and ■ refer to the cases $h = 0, 0.4, 0.8$.

Solid curves are the random walk prediction, open disks represent the Mean Field prediction.

# Results: $\delta = 1$



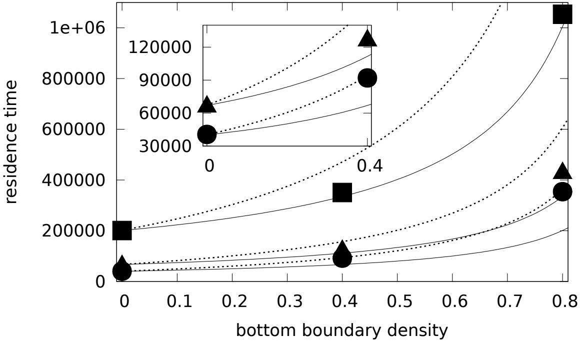

Parameters: $h = 0.4$, and $L_1 = 100$; •, ▲ and ■ refer to $\varrho_{\rm d} = 0, 0.4, 0.8$.

Solid curves and open disks represent the BD and the MF predictions.

Open squares denotes the MF and BD result for $L_2$ large:

$$R^{\rm MF} = R^{\rm BD} \approx \frac{L_2}{(1-h)(1-\bar{\varrho})\delta}$$

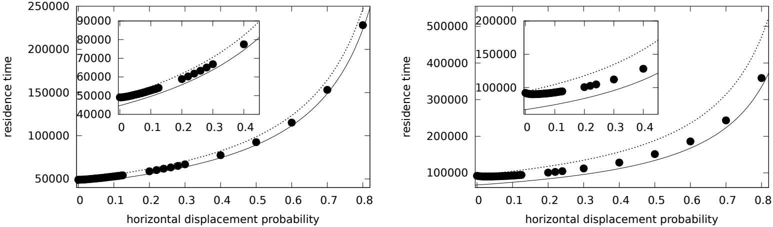
# Results: $\delta = 0$ e $\varrho_{\mathrm{d}} > 0$




Parameters: $\delta = 0$, $L_1 = 100$, and $L_2 = 200$; the symbols •, ▲ and ■ refer to the cases $h = 0, 0.4, 0.8$.

Solid curves are the random walk prediction, dashed lines represent the Mean Field prediction.

# Transition between the MF and BD regimes

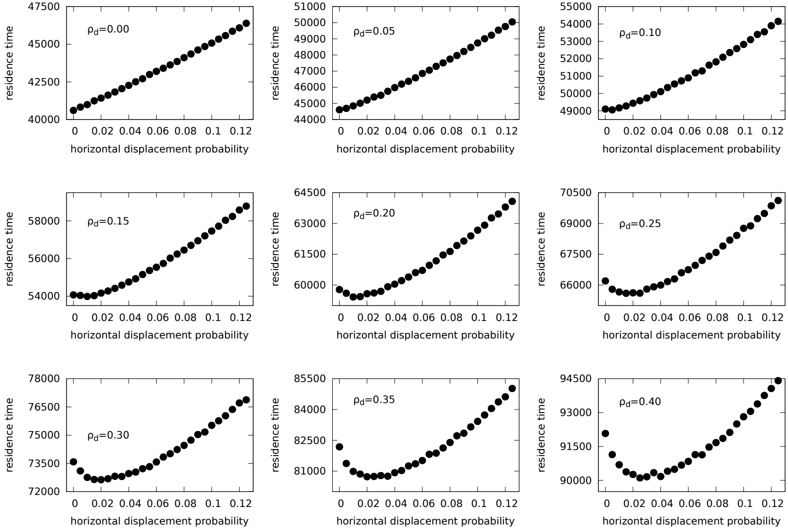



Residence time vs. horizontal displacement probability $h$ for $L_1 = 100$, $L_2 = 200$, $\delta = 0$, $\varrho_{\rm d} = 0.1$ (left) and $\varrho_{\rm d} = 0.4$ (right). Solid curves are the random walk prediction, dashed lines represent the Mean Field prediction.

Explicit formulas

$$R^{\rm MF} = \frac{1}{1-h}\,\frac{1+\varrho_{\rm d}}{1-\varrho_{\rm d}}(L_2+1)^2 \;\;\text{and}\;\; R^{\rm BD} = \frac{1}{1-h}\,\frac{1}{1-\varrho_{\rm d}}L_2(L_2+1)$$

# Non–monotonicity on $h$: $\delta = 0$, $L_1 = 100$, and $L_2 = 200$



# Strip with obstacles

# Mean Field stationary state

Solve the equation

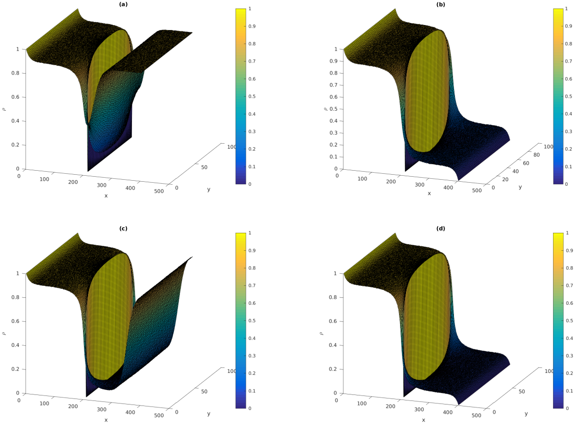
$$\frac{1}{2}h\frac{\partial^2\varrho}{\partial y^2} + \frac{1}{2}(1-h)\frac{\partial^2\varrho}{\partial x^2} - \delta(1-h)\frac{\partial}{\partial x}[\varrho(1-\varrho)] = 0$$

in the domain equal to the strip minus the region occupied by the obstacle

- with Neumann homogeneous boundary conditions on the vertical boundaries and on the boundaries of the obstacle
- with Dirichlet boundary conditions $\varrho_{\mathrm{u}}$ and $\varrho_{\mathrm{d}}$ on the top and on the bottom boundaries.

Remark: the solution will depend on the horizontal coordinate $y$.

# Mean Field stationary profile




Parameters: $\rho_\mathrm{d} = 0.9$ (left), $\rho_\mathrm{d} = 0.0$ (right), $W = 85$ (top), $W = 90$ (bottom), $L_1 = 100$, $L_2 = 400$, $h = 0.5$, $\delta = 0.05$, $\rho_\mathrm{u} = 1$, $O_2 = 3$.

# Residence time estimate

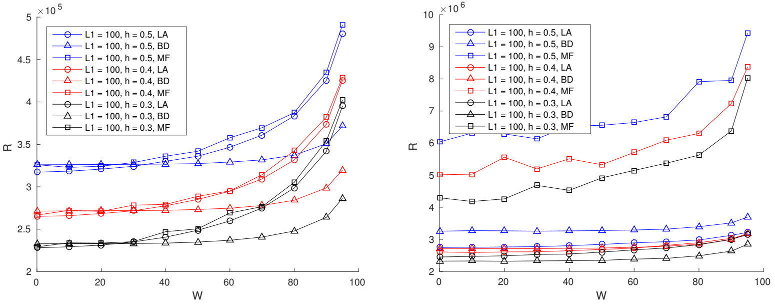
As before we compare the Monte Carlo (labelled LA), the Mean Field (MF) and the Birth–and–Death (BD).

We use the horizontally averaged density profile

$$\varrho(x) = \frac{1}{L_1} \sum_{x=1}^{L_1} \varrho(y, x)$$

We do not expect quantitative agreement, since in this case the horizontal translation invariance is lost.

# Results: zero drift ($\delta = 0$)

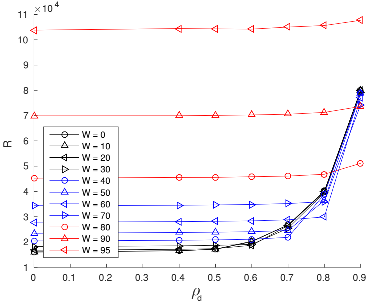



Parameters: $L_1 = 100$, $L_2 = 400$, $O_2 = 3$, $h = 0.5$, $\rho_{\mathrm{d}} = 0$ (left), and $\rho_{\mathrm{d}} = 0.9$ (right).

We find the expected results: increasing with $W$, good agreement on the left for $W$ small enough (MF much better than BD), poor agreement on the right (BF better that MF).

# Results: not zero drift ($\delta > 0$)

Parameters: $L_1 = 100$, $L_2 = 400$, $h = 0.5$, $O_2 = 3$, and $\delta = 0.1$.

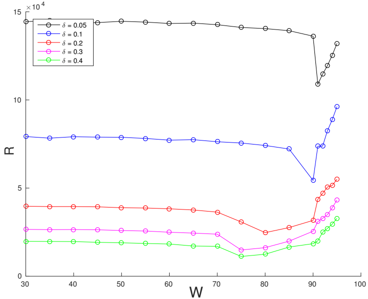



Remarks:

- good agreement with MF (not reported on the picture);
- increasing with $W$ for $\varrho_{\rm d} \leq 0.5$;
- not monotonic behavior with $W$ for $\varrho_{\rm d} > 0.5$;

# Focus on the case $\varrho_\mathrm{d} = 0.9$

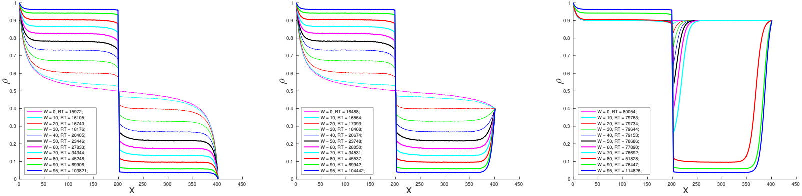
Parameters: $L_1 = 100$, $L_2 = 400$, $h = 0.5$, $O_2 = 3$, and $\varrho_\mathrm{d} = 0.9$ (right).




The residence time $R$ is almost constant till a critical value of $W$ is reached, where it decreases abruptly. After that it starts to increase.

We can explain this effect in terms of the occupation number stationary profile.

# Horizontally averaged occupation number profile




Parameters: lattice $100 \times 400$, $h = 0.5$, $\delta = 0.1$, $\rho_{\mathrm{d}} = 0$ (left), $\rho_{\mathrm{d}} = 0.4$ (center), $\rho_{\mathrm{d}} = 0.9$ (right), and $O_2 = 3$. Remarks:

- left and center pictures are similar: nothing happens for $\varrho_{\mathrm{d}} \leq 0.5$;
- at $\varrho_{\mathrm{d}} = 0.9$ the average occupation number in the upper region is much higher and hence so it is the residence time;
- for $W = 80$ there is an abrupt change in the profile after the obstacle: the residence time decrease;
- the further increase of the occupation number in the upper part justifies the finale increase of the residence time.

# Comments

- modeled the problem by means of a lattice exclusion process
- developed two analytical tools to estimate the residence time
- without obstacles: we found strange behaviors in the zero drift regime when particles are allowed to enter through the bottom boundary
- with obstacles: in the not zero drift regime we found a not monotonic behavior of the residence time with respect to the width of the obstacle
- explored the connection of such a behavior with the shape of the occupation number stationary profile
- what is the reason of the behaviors described above?
- what happens if other parameters of the obstacles are changed? Position? Height?
- last question is connect to Alessandro Ciallella's poster: a similar problem in the framework of the Lorentz Gas system

# Addenda